Math riddle

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Riddle

Imagine you have a bucket of water containing two distinct types of bacteria: A and B. You start with one bacterium of each type, A and B. Every time you introduce a particle of food into the bucket, one of the bacteria consumes it. The bacterium that eats the food reproduces, adding one more bacterium of its type to the population. If you were to drop 100 particles of food into the bucket, can you predict the distribution of the A-type bacteria?

Example

Suppose the bucket contains 3 A-type bacteria and 2 B-type bacteria. If you drop a food particle and an A-type bacterium consumes it, the population changes to 4 A-type bacteria and 2 B-type bacteria. On the other hand, if a B-type bacterium eats the food, the population becomes 3 A-type and 3 B-type bacteria.

Solution

Let's denote the number of A-type bacteria as a and B-type bacteria as b.

Base Case

For $(n = 1)$ we have a uniform distribution

$$P(a = 1) = \frac{1}{2},$$
$$P(a = 2) = \frac{1}{2}.$$

Inductive Hypothesis

Assume that for k particles of food, the distribution of the number of A-type bacteria is uniform. That is, for each possible a,

$$P(a = i) = \frac{1}{k+1}$$

for i from 1 to $(k+1)$.

Inductive Step

After dropping the $(k+1)$th particle, the bucket could have anywhere from $a = 1$ to $a = k+2$ bacteria of type A. We aim to show that

$$P(a = i) = \frac{1}{k+2}$$

for each i in this range.

For any configuration with i bacteria of type A after k particles, the probability that it transitions to $(i+1)$ bacteria of type A after the $(k+1)$th particle is $\frac{i}{k+1}$. Given our inductive hypothesis, the probability of being in this configuration after k particles is $\frac{1}{k+1}$. Thus, the probability of transitioning to $(i+1)$ bacteria of type A from this configuration is:

$$\frac{i}{(k+1)^2}$$

Likewise, the chance of transitioning to i bacteria of type A from a configuration with $(i-1)$ bacteria of type A is:

$$\frac{k+2-i}{(k+1)^2}$$

Considering these probabilities for all possible transitions, we obtain:

$$\sum_{i=1}^{k+1} \left(\frac{i}{(k+1)^2} + \frac{k+2-i}{(k+1)^2} \right)$$

For every term i in the summation, we always have:

$$\frac{i + (k+2-i)}{(k+1)^2} = \frac{k+2}{(k+1)^2}$$

Given there are $(k+1)$ terms in the summation, and the probabilities must sum to 1, the individual probability for each state i is:

$$\frac{1}{k+2}$$

This confirms the uniform distribution for the $(k+1)$th particle. By the principle of mathematical induction, if the distribution of the number of A-type bacteria is uniform after n particles, it remains uniform after $(n+1)$ particles. Thus, for any number of food particles, the distribution of A-type bacteria is uniform.